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RELATIONS BETWEEN RESPONSE AND FOURIER SPECTRA OF SHOCK FUNCTIONS

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Abstract—Using the running Fourier transform technique for solving the equation of motion it is found that velocity and pseudo-velocity response spectra of shock functions are bounded above and below by the amplitude and Fourier spectra respectively. The gap between bounds and spectra is assessed in a numerical example.

INTRODUCTION

A FUNCTION $s(t)$ defined in $-\infty < t < \infty$ by

$$s(t) = \begin{cases} 0 & t < a \\ f(t) & a \leq t \leq b \\ 0 & t > b, \end{cases} \tag{1}$$

where $f(t)$ is integrable over (a, b) in the Riemann sense, is referred to herein as a shock function. The intervals $t < a$, $a \leq t \leq b$, $t > b$ indicate respectively the initial, forced and free eras of the shock function. Shock functions are an extension of the class of functions called simple pulses [1]. The facts justifying the definition of shock function are, first, that a great majority of forcing functions considered in shock do satisfy definition (1), and, second, that important properties of these forcing functions are just consequences of this definition.

It is evident, for instance, that any shock function possesses a Fourier transform. The Fourier transform of a shock function s is denoted herein by

$$F_s(u) = \int_{-\infty}^{\infty} s(\tau)e^{-iu\tau}\,d\tau. \tag{2}$$

It is also evident that any shock function has a running Fourier transform [2]. The running Fourier transform of a shock function is equal to

$$F_s(t, u) = \int_{-\infty}^{t} s(\tau)e^{-iu\tau}\,d\tau. \tag{3}$$

The Fourier spectrum and running Fourier spectrum are defined respectively as the absolute value of the corresponding transforms.

Consider a linear single degree of freedom system which equation of motion is

$$\ddot{x} + 2\lambda\omega\dot{x} + \omega^2 x = s. \tag{4}$$

Here s is a shock function, $\omega^2 > 0$, $\lambda \geq 0$ and the initial conditions are homogeneous and given at time $t_0 \leq a$, namely,

$$x(t_0) = x_0 = 0 \tag{5a}$$

$$\dot{x}(t_0) = v_0 = 0. \tag{5b}$$

It is clear that for any shock function equation (4) with initial conditions (5) has a solution which can be expressed in terms of the well known convolution integral

$$x(t, \omega) = \int_{t_0}^{t} s(\tau)h(t-\tau)\,\mathrm{d}\tau \tag{6}$$

where $h(t)$ is the unit-impulse response of equation (4).

The displacement, velocity, pseudo-velocity and pseudo-acceleration response spectra of s are defined respectively by

$$S_{x,s}(\omega) = \sup_t |x(t, \omega)| \tag{7}$$

$$S_{\dot{x},s}(\omega) = \sup_t |\dot{x}(t, \omega)| \tag{8}$$

$$S_{\bar{\dot{x}},s}(\omega) = \omega S_{x,s}(\omega) \tag{9}$$

$$S_{\bar{a},s}(\omega) = \omega^2 S_{x,s}(\omega). \tag{10}$$

Existance, uniqueness, boundedness and continuity properties of the functions defined by equations (2), (3) and (6)–(10) are direct consequences of the definition of shock function.

In this article the relations existing between response and Fourier spectra are derived in the context of shock functions. For this purpose it is particularly indicated to solve the equation of motion using the running Fourier transform. Apparently the running Fourier transform has not been used for solving differential equations. A novel feature of this technique is that instead of the inversion step, required by ordinary Fourier or Laplace transforms, which reduces the problem to contour integration, the running Fourier transform requires solving a system of simultaneous linear algebraic equations. This study is restricted to relations between Fourier and response spectra in undamped linear single-degree-of-freedom systems. Therefore undamped response spectra are referred to herein shortly as response spectra.

RELATIONS BETWEEN FOURIER SPECTRA AND THE SOLUTION OF THE EQUATION OF MOTION

If λ is set to be equal to zero, equation (4) is the equation of motion of an undamped linear single-degree-of-freedom system. Applying definition (3) to equation (4) gives

$$\int_{t_0}^{t} \ddot{x}\mathrm{e}^{-iu\tau}\,\mathrm{d}\tau + \omega^2 \int_{t_0}^{t} x\mathrm{e}^{-iu\tau}\,\mathrm{d}\tau = \int_{t_0}^{t} s\mathrm{e}^{-iu\tau}\,\mathrm{d}\tau = F_s(t, u). \tag{11}$$

Integrating by parts and using initial conditions (5)

$$(\dot{x} + iux)\mathrm{e}^{-iut} + (\omega^2 - u^2)\int_{t_0}^{t} x\mathrm{e}^{-iu\tau}\,\mathrm{d}\tau = F_s(t, u). \tag{12}$$

Substitute roots of equation $u^2 - \omega^2 = 0$ into equation (12)

$$\dot{x} + i\omega x = e^{-i\omega t} F_s(t, \omega) \tag{13a}$$

$$\dot{x} - i\omega x = e^{-i\omega t} F_s(t, -\omega). \tag{13b}$$

This is a system of two simultaneous linear equations in the unknowns x and $\dot{x}$ having a non zero determinant. Thus x and $\dot{x}$ are uniquely determined.

Multiply equation (13a) by equation (13b) and divide the result by 2

$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2 = \tfrac{1}{2}|F_s(t, \omega)|^2. \tag{14}$$

Equation (14) establishes a relation between the running Fourier spectrum of s and the solution of equation (4). For a fixed mechanical oscillator with unit mass, equation (14) is a statement of the principle of conservation of mechanical energy in which the total energy appears as an integral of motion. The sum of potential and kinetic energy is equal to half times the square of the running Fourier spectrum. Obviously, the right hand side of equation (14) is equal to the work done by the external force s from time $t = -\infty$ to time t.

For $t \geq b$ it follows from definitions (1)–(3) that

$$F_s(t, \omega) = F_s(\omega), \qquad t \geq b. \tag{15}$$

Thus, for $t \geq b$ equations (14) and (15) give

$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2 = \tfrac{1}{2}|F_s(\omega)|^2, \qquad t \geq b. \tag{16}$$

Equation (16) states that the total energy of the oscillator in the free era is equal to half times the square of the Fourier spectrum of s.

AMPLITUDE SPECTRUM

The running Fourier transform defined by equation (3) can be written as

$$F_s(t, \omega) = A_s(t, \omega) e^{-i\phi_s(t,\omega)} \tag{17a}$$

where

$$A_s(t, \omega) = |F_s(t, \omega)| \tag{17b}$$

and

$$\phi_s(t, \omega) = \tan^{-1}\left\{-\frac{\operatorname{Im} F_s(t, \omega)}{\operatorname{Re} F_s(t, \omega)}\right\}. \tag{17c}$$

$A_s(t, \omega)$ and $\phi_s(t, \omega)$ are referred to respectively as the running amplitude and the running phase of the shock function s.

The amplitude response spectrum of a shock function is defined herein as

$$S_{A,s}(\omega) = \sup_t A_s(t, \omega). \tag{18}$$

In account of equations (14), (17b) and (18), $\tfrac{1}{2}S_{A,s}^2(\omega)$ is equal to the maximum energy per unit of mass delivered by the shock function into a fixed oscillator of frequency ω, whereas from equations (8), (9) and (14) $\tfrac{1}{2}S_{\dot{x},s}^2(\omega)$ and $\tfrac{1}{2}S_{\dot{x},s}^2(\omega)$ are respectively equal to the maximum kinetic and maximum potential energy per unit of mass delivered by s into the same oscillator.

It is interesting to notice that the amplitude spectrum of any shock function is bounded for any frequency including the limit values $\omega = 0$ and $\omega = \infty$. This means that any shock function delivers a finite amount of energy into any single-degree-of-freedom system including the limit, but possible, systems consisting of a single mass or a single spring. It is also a property of shock functions that boundedness of the pseudo-acceleration spectrum for any frequency, including the referred limit systems, insures the force in the spring to be always finite.

UPPER AND LOWER BOUNDS FOR RESPONSE SPECTRA

A. *Upper bounds*

Replacing equation (17a) into (13a) and separating real and imaginary parts

$$\dot{x} = A_s(t, \omega)\cos[\omega t - \phi_s(t, \omega)] \tag{19a}$$

$$\omega x = A_s(t, \omega)\sin[\omega t - \phi_s(t, \omega)]. \tag{19b}$$

From equations (8), (18), (19a) and equations (9), (18), (19b) it follows

$$S_{\dot{x},s}(\omega) = \sup_t |\dot{x}| \le \sup_t A_s(t, \omega) = S_{A,s}(\omega) \tag{20a}$$

$$S_{\bar{\dot{x}},s}(\omega) = \sup_t |\omega x| \le \sup_t A_s(t, \omega) = S_{A,s}(\omega). \tag{20b}$$

Equations (20) show that velocity and pseudo-velocity spectra are bounded above by the amplitude spectrum.

B. *Lower bounds*

Consider equations (19) for $t \ge b$. In account of equations (1), (3), (15) and (17) equations (19) can be written as

$$\dot{x} = |F_s(\omega)|\cos[\omega t - \phi_s(b, \omega)], \qquad t \ge b \tag{21a}$$

$$\omega x = |F_s(\omega)|\sin[\omega t - \phi_s(b, \omega)], \qquad t \ge b. \tag{21b}$$

From equations (8), (21a) and (9), (21b) it follows:

$$S_{\bar{\dot{x}},s}(\omega) = \sup_t |\dot{x}| \ge \sup_{t \ge b} |\dot{x}| = |F_s(\omega)| \tag{22a}$$

$$S_{\bar{\dot{x}},s}(\omega) = \sup_t |\omega x| \ge \sup_{t \ge b} |\omega x| = |F_s(\omega)|. \tag{22b}$$

Equations (22) show that velocity and pseudo-velocity spectra are bounded below by the Fourier spectrum of s.

From relations (20) and (22) it follows that if

$$|F_s(\omega)| = S_{A,s}(\omega) \tag{23}$$

then

$$S_{\dot{x},s}(\omega) = S_{\bar{\dot{x}},s}(\omega). \tag{24}$$

Therefore, equality between Fourier and amplitude spectra is a sufficient condition for equality of velocity and pseudo-velocity spectra.

ILLUSTRATIVE NUMERICAL EXAMPLE

This numerical example was calculated in a digital computer using a program written by the author. The purpose was to obtain a numerical estimate in a particular case on how close to velocity and pseudo-velocity spectra were the values of the upper and lower bounds given by equations (20) and (22). The chosen function shown in Fig. 1 is a portion of the S80E component of the ground acceleration recorded at Golden Gate Park in the earthquake of 22 March 1957. The amplitude, velocity, pseudo-velocity and Fourier spectra are given in Fig. 2. The upper curve is the amplitude spectrum and the lower curve is the Fourier spectrum.

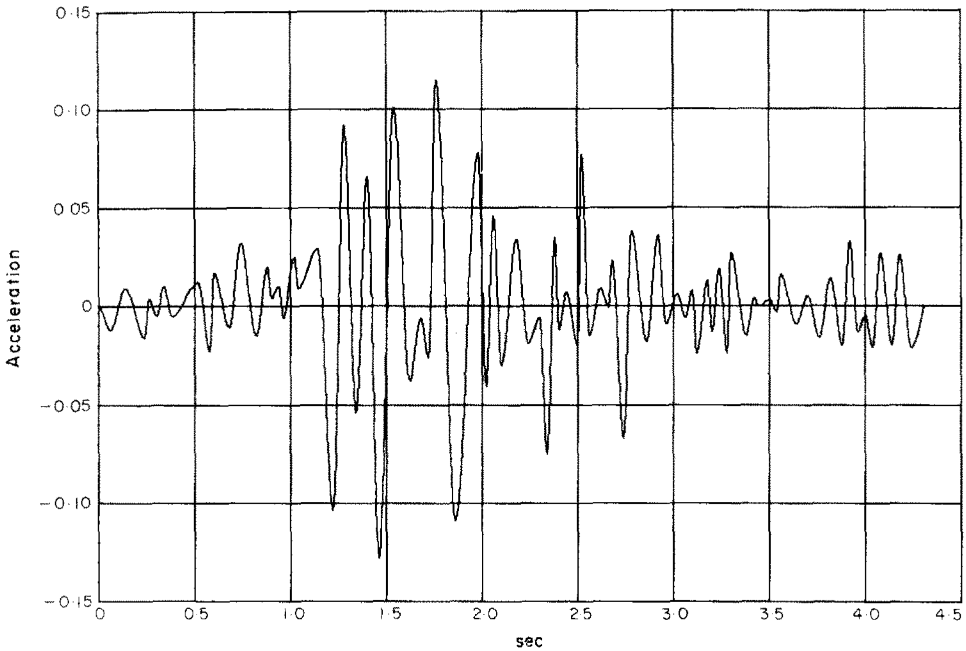

FIG. 1. San Franciso Earthquake of 22 March 1957 recorded at Golden Gate Park component S80E.

Results of this example can be summarized as follows: The upper bound shows consistently very close values to both velocity and pseudo-velocity spectra for all frequencies with the exception of frequencies below 12 rad./sec. In this later interval the upper bound remains very close to the velocity spectrum only. The lower bound is close to velocity or pseudo-velocity spectra only in some selected frequency intervals. For the remaining frequencies the lower bound differs considerably from the other spectra. At $\omega = 53$ rad./sec, for instance, the lower bound is less than $\frac{1}{8}$ the values of response spectra.

CONCLUSIONS

Relations between response and Fourier spectra are investigated using the running Fourier transform for solving the equation of motion. A novel feature of this technique is that the problem of solving a differential equation is reduced to the solution of a system

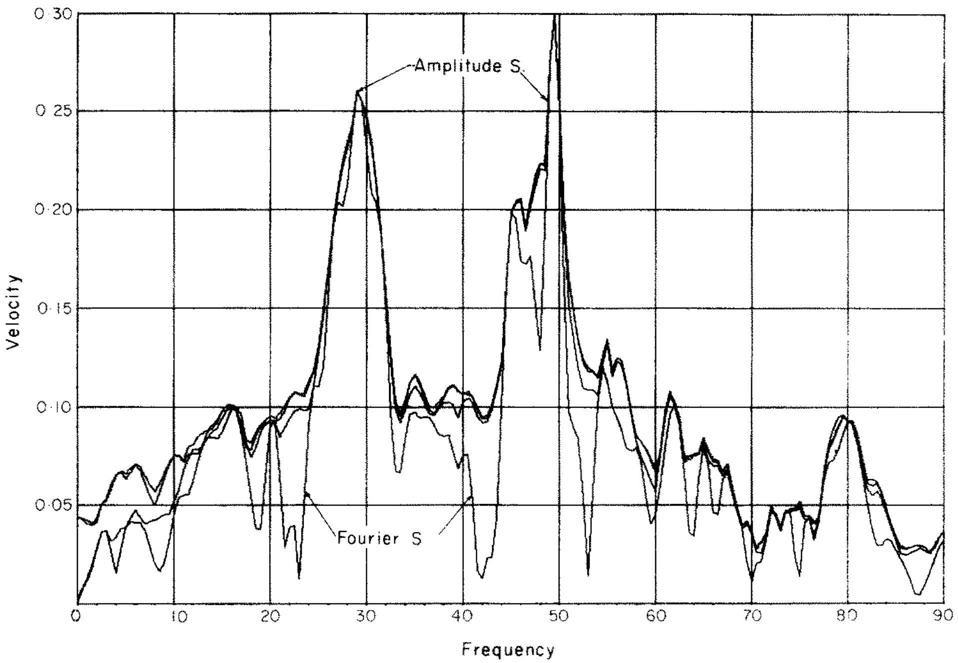

FIG. 2. Velocity and pseudo-velocity spectra bounded above and below by the amplitude and Fourier spectra.

of simultaneous linear equations, instead of the inversion step required by ordinary Fourier or Laplace transforms. This technique is particularly indicated herein for establishing a relation between the running Fourier spectrum and the solution of the equation of motion. Through the running Fourier spectrum the amplitude spectrum is defined. The amplitude spectrum is a measure of the maximum energy which a shock function can deliver into a fixed linear undamped single-degree-of-freedom system.

Upper and lower bounds for response spectra are given, namely, velocity and pseudo-velocity spectra are bounded above and below by the amplitude and Fourier spectra respectively. A numerical example indicates, for the particular shock function of an earthquake acceleration, that the upper bound provided by the amplitude spectrum is considerably closer to velocity and pseudo-velocity spectra as compared with the lower bound given by the Fourier spectrum.

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REFERENCES

[1] Y. C. FUNG, Shock loading and response spectra, *Colloq. Shock and Structural Response*. ASME (1960).
[2] C. H. PAGE, Instantaneous power spectra. *J. appl. Phys.* **23**, 103 (1952).

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Абстракт—Используя метод переменного преобразования Фурие для решения уравнения движения оказывается, что спектры характеристик скорости и псевдо-скорости функций удара ограничены оверху и снизу, соответстненно, амплитудой и спектрами Фурье. На численном примере определяется интервал между пределами и опектрами.